Optimal Pricing for Improving Efficiency of Taxi Systems

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Abstract

In Beijing, most taxi drivers intentionally avoid working during peak hours despite of the huge customer demand within these peak periods. This dilemma is mainly due to the fact that taxi drivers' congestion costs are not reflected in the current taxi fare structure. To resolve this problem, we propose a new pricing scheme to provide taxi drivers with extra incentives to work during peak hours. This differs from previous studies of taxi market by considering market variance over multiple periods, taxi drivers' profit-driven decisions, and their scheduling constraints regarding the interdependence among different periods. The major challenge of this research is the computational intensiveness to identify optimal strategy due to the exponentially large size of a taxi driver's strategy space and the scheduling constraints. We develop an atom schedule method to overcome these issues. It reduces the magnitude of the problem while satisfying the constraints to filter out infeasible pure strategies. Simulation results based on real data show the effectiveness of the proposed methods, which opens up a new door to improving the efficiency of taxi market in megacities (e.g., Beijing).

1 Introduction

As an efficient and convenient way for short-distance travel, taxi has been an indispensable mode of transport in most large cities. It is easily accessible and comfortable compared with other types of public transport. Therefore, it is essential for decision-makers to regulate the taxi market to balance taxi supply and customer demand, especially during peak hours. However, this has been a challenging issue due to the decentralized nature of the taxi systems and taxi drivers’ profit-driven behavior.

A long-standing problem in the taxi market in Beijing is the insufficient taxi supply during peak hours. During these periods most taxi drivers intentionally avoid working regardless of the huge demand (Fig. 1(a)). As a result, a large number of customers have to spend a lot of time (sometimes more than 2 hours) waiting for a taxi (Fig. 1(b)). Some even may have to switch to unlicensed taxis, which typically charge a higher rate and pose threats to road safety. The improper pricing of the current taxi fare structure, as revealed by recent interviews to taxi drivers is the main cause of this dilemma. The taxi fare in Beijing is calculated primarily by trip distance, and it stays stable throughout the day. Typically, travel speed is low during peak hours due to traffic congestion. As a result, a taxi driver may choose not to work since the income may be even lower than the operation cost (gas cost and etc.).

There has been some research works on analyzing taxi market equilibrium in the literature, but market efficiency is ignored by them. Yang et al. [2005a] studied the taxi market in the presence of traffic congestion with the objective of identifying market equilibrium. The study did not consider taxi drivers’ decision making process (i.e., work or not). Other than that, the number of working taxis was set to a constant value instead of a variable which is determined by taxi drivers’ operation strategies. In another study, Yang et al. [2005b] analyzed the taxi market with consideration of market variance over multiple periods. This study provided a framework for modeling the taxi market under a multi-period scenario. However, as key factors that affect taxi drivers’ decision, variance of traffic condition and interdependence of each period are not revealed in this study. Recently, there are some works on traffic control and intersection management using AI techniques [Au et al., 2011; Bazzan, 2009; Dresner and Stone, 2008; Xie et al., 2012], but these approaches cannot be used to improve taxi market efficiency. In this paper, we propose to adjust the current taxi fare to give taxi drivers extra incentives for working in peak hours, which has been adopted in several countries like Singapore. How-
ever, to the best of our knowledge, there has been no research on how to generate the optimal fare structure.

To compute the optimal fare, we first analyze how taxi drivers react to changing market conditions and how these reactions affect market conditions in return. Different from previous studies, we consider the variations of market conditions over multiple time periods since taxi drivers’ strategies in one period are highly dependent on strategies in other periods due to scheduling constraints on total working time and maximum continuous working time. Then we calculate the taxi drivers’ best-mixed strategy to study their behavior in response to market condition variations. However, a driver’s strategy space increases exponentially with the number of working periods. To resolve this scalability issue, an atom search the optimal fare that maximizes the overall efficiency of the system. Specially, customer demand in period $i$ is defined as

$$D^i = D_{\text{max}} \exp\{-\beta(F^i/m + \phi_1 L^i + \phi_2 W^i)\}, \quad(4)$$

where $D_{\text{max}}$ is a constant representing the maximal potential customer demand. $F^i$, $L^i$ and $W^i$ are trip fare, travel time, and customer waiting time respectively. $m$ is the average number of passengers in a trip, $\phi_1 > 0$ and $\phi_2 > 0$ are parameters used for converting $L^i$ and $W^i$ into monetary costs respectively. $\beta > 0$ is a demand sensitivity parameter.

We define $L^i$ as $L^i = \bar{d}/V^i$, and $F^i$ as $F^i = F^{i\prime}(f^i, \bar{d})$, where $\bar{d}$ denotes average trip distance. We assume that $\bar{d}$ is a constant, and denote $F^{i\prime}$ as $F^{i\prime}(f^{i\prime})$ for short. Waiting time $V^i$ is negatively related to the number of vacant taxis available:

$$V^i = \frac{\omega}{N^i(1 - \frac{L^i}{\bar{d}} \cdot \frac{D^i}{mN^i})} = \frac{\omega}{N^i - D^i d^i/(mV^i\bar{d})}, \quad(5)$$

where $1 - L^i/\bar{d} \cdot D^i/mN^i$ represents a taxi’s vacancy rate.

2.2 Customer Demand

We assume that customer demand in a specific period depends on monetary and time cost for taxi service, i.e., taxi fare, travel time, and waiting time, as in many existing works such as [Yang et al., 2002; 2005b]. Here customer demand represents the number of customers transported by the taxi system. Specially, customer demand in period $i$ is defined as

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where $D_{\text{max}}$ is a constant representing the maximal potential customer demand. $F^i$, $L^i$ and $W^i$ are trip fare, travel time, and customer waiting time respectively. $m$ is the average number of passengers in a trip, $\phi_1 > 0$ and $\phi_2 > 0$ are parameters used for converting $L^i$ and $W^i$ into monetary costs respectively. $\beta > 0$ is a demand sensitivity parameter.

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2.3 Taxi Drivers’ Strategy

To simplify the analysis, we assume that all taxi drivers are identical and they adopt the same operation strategy, i.e., in period $i$ they all choose to work with probability $p^i$, and not to work with probability $1 - p^i$. Given the total number of licensed taxis $N_t$, $N_{t^i} = N_t \cdot p^i$ taxis choose to work. Substitute this into Eq. (2), $V^i$ can be written as a function of $p^i$, $V^i = V^{i\prime}(p^i)$ for short.

Taxi drivers are self-interested and always adjust their working probability $p^i$ to maximize their individual revenue. We define a taxi driver’s utility as

$$U^i = p^i \cdot (\frac{D^i}{mN_{t^i}} \cdot F^i - c_g \cdot \tau) = \frac{D^i}{mN_t} F^i - p^i c_g \tau, \quad(6)$$

where $\frac{D^i}{mN_{t^i}} \cdot F^i - c_g \cdot \tau$ represents the revenue of working in period $i$, as $\frac{D^i}{mN_{t^i}}$ is the number of trips a taxi serves, and $c_g$ is the cost on gasoline consumption per unit time.

1We choose to improve the current fare structure, instead of re-designing a new fare structure, or adopting other existing structures. This is because 1) other fare structures, such as the time-dependent structure which is widely used in some European countries, may not be applicable to Beijing due to differences in culture, economy, and most importantly the level of traffic congestions in comparison to other cities; 2) our idea of giving taxi drivers’ extra incentives can be well achieved with the current fare structure. In addition, our model can also be easily extended to other fare structures if necessary.
From Eqs. (4) and (5), $D_i = D_i(W_i(D_i, p^i), p^i, f^i)$. Given $p^i$ and $f^i$, $D_i$ is in fact a function of itself. We define the following equation

$$\Phi(D^i) = D^i - D^i(W^i(D^i, p^i), p^i, f^i) = 0.$$ 

(7)

Yang et al. [2005b] proved that there is one and only one solution to this equation. From Eq. (5) and the fact that $W^i(i) > 0$, we have $0 < D^i < mN_w V^i \tau / d^i$. Thus, $\Phi(D^i)$ is a continuous function of $D^i \in (0, mN_w V^i \tau / d^i)$, and

$$\Phi(D^i) = 1 - \frac{\partial D^i}{\partial W^i} \frac{\partial W^i}{\partial D^i} = 1 + \frac{\partial \Phi(D^i)}{\partial mV^i \tau} \cdot D^i(W^i)^2 > 1.$$ 

Furthermore, it can be easily verified that

$$\lim_{D^i \to 0} \Phi(D^i) = -D^i(W^i(0)) < 0,$$

$$\lim_{D^i \to mN_w V^i \tau / d^i} \Phi(D^i) = mN_w V^i \tau / d^i > 0.$$ 

Thus, there is one and only one point where $\Phi(D^i) = 0$, and correspondingly one and only one solution to this equation.

By substituting this solution into Eqs. (5) and (6), we could calculate the values of $W^i$ and $U^i$. Therefore, $D^i, W^i$ and $U^i$ are all implicitly and uniquely determined by $p^i$ and $f^i$. We write $U^i$ as a function of $U^i = U^i(p^i, f^i)$. Given $f^i$, taxi drivers would maximize their utility by choosing an optimal working probability $p^*_i$, such that

$$p^*_i = \arg \max_{p^i \in [0,1]} U^i(p^i, f^i).$$ 

(8)

This paper assumes that each taxi driver chooses his optimal working probability to maximize his revenue. Taxi drivers’ working probability affects traffic condition, waiting time, and customer demand. Since we do not distinguish different taxi drivers, the optimal working probability for a given taxi is in fact an equilibrium strategy for all drivers.

2.4 Computing the Optimal Price

Our objective is to maximize taxi system efficiency, which is measured by customer demand $D^i$ under market equilibrium. Intuitively, given our model, higher customer demand indicates higher driver working probability, higher driver revenue, and higher social welfare. Moreover, our framework can also be easily extended to optimize other objective functions. We construct an optimization program for period $i$ as follows:

$$\max_{p^i, f^i} D^i,$$ 

(9)

$$\text{s.t. } \begin{cases} p^i \in \arg \max_{p^i \in [0,1]} U^i(p^i, f^i), \\ D^i, W^i, V^i \geq 0. \end{cases}$$ 

(10)

The above program is a bilevel program with two levels involved: an upper level defined in Eq. (9), and a lower level defined in the first constraint. To look into the lower level program, we consider a period from 17:00 to 18:00 using real data (Section 4.1). From the plot of $U^i$ with different $p^i$, we find that for $f^i = \text{Y}2.00$ to $\text{Y}5.00$, when $p^i$ is very close to 0, $U^i$ first drops for a short section (Fig. 2(b)). This is because the number of taxis increases by more than the increment of customer demands. However, this situation happens with an extremely small percentage (from Fig. 2(b) less than 0.4%) of working taxis, which is unlikely to happen in reality. From Fig. 2(a), we see that $U^i$ is almost a concave function of $p^i$, which indicates that there is always only one optimal solution to the lower level problem.

Moreover, it is unnecessary to consider extremely large or small values for $f^i$, since drivers cannot make money with very low price, and the demand would be extremely small with a very high price. It is also unnecessary to consider prices with too many digits, such as $\text{Y}2.345$. Thus we focus on some candidate prices within a reasonable range $[f_{\text{min}}, f_{\text{max}}]$ (e.g., $\text{Y}1.00, \text{Y}1.20, \text{Y}1.40, ..., \text{Y}5.00$). Alg. 1 searches the optimal price from several candidate prices within $[f_{\text{min}}, f_{\text{max}}]$ with the difference $\Delta$. Line 5 solves the lower level optimization program redefined as:

$$\max_{p^i, D^i, W^i} U^i = \frac{D^i F^i}{m N_i} - p^i c_g \tau,$$ 

(11)

$$\text{s.t. } \begin{cases} D^i - D^i_{\text{max}} e^{-\beta (F^i / m + \phi_1 L^i + \phi_2 W^i)} = 0, \\ W^i - \omega (N_i - D^i d^i / m V^i \tau) = 0, \\ 0 \leq p^i \leq 1, 0 < D^i, W^i. \end{cases}$$ 

(12)

Note that since $U^i$ as a function of $p^i$ and $f^i$ is not defined explicitly, we take $D^i$ and $W^i$ as independent variables, and consider their dependence in the first two constraints.

3 Optimal Pricing for Multiple Periods

In this section, we extend the modeling framework for period $i$ to a whole day, which is divided into $n$ periods. We extend each period-dependent variable to a vector for multiple periods, e.g., we denote average travel speed as $V = \text{...}$
Similarly, we denote customer demand, customer waiting time and each taxi driver’s utility as $D$, $W$ and $U$ respectively. We denote the percentage of working taxis and per unit distance price as $p$ and $f$ respectively.

### 3.1 Taxi Driver’s Optimal Mixed Strategy

For multiple periods, a taxi driver’s pure strategy corresponds to an operation schedule over all periods. For example, for $n = 3$, a pure strategy could be: work in the first and the third periods, and rest in the second period. We denote such a pure strategy by a vector $s = (s_1, ..., s_n)$, such that $s_i = 1$ if the driver chooses to work in period $i$, and $s_i = 0$ otherwise. Moreover, each taxi driver faces additional realistic scheduling constraints that must be taken into account:

- **C1:** $\sum_{i=1}^{n} s_i \leq n_w$. C1 requires that each taxi driver should not work for more than $n_w$ periods a day.
- **C2:** $\max_{i,j \in \mathbb{N}, 1 \leq j \leq n, s_k = 1} (j - i) \leq n_e$. C2 requires that each taxi driver should not work continuously for more than $n_e$ periods.

Let the set of feasible pure strategies be $S$. A mixed strategy can be represented as a vector $x$, whose element $x_s$ is the probability assigned to schedule $s$. The mixed strategy space is $X = \{x : 0 \leq x_s \leq 1, \sum_{s \in S} x_s = 1\}$. A mixed strategy determines the percentages $p$ of working taxis in each period:

$$p^i = \sum_{s \in S} x_s \cdot s_i,$$  \hspace{1cm} (13)

Therefore, $p$ is a function of $x$. According to analysis in Section 2.3, once $p$ and $f$ is specified, travel speed, customer demand, waiting time and taxi drivers’ utility for each period are all determined. We define taxi drivers’ overall utility $U_{mp}$ as $U_{mp} = \sum_{i=1}^{n} U^i$, which is determined by $p$ and $f$. We can thus write $U_{mp} = U_{mp}(p, f) = U_{mp}(x, f)$, since $p$ is determined by $x$. Thus a driver’s optimal strategy is

$$x^* \in \arg \max_{x \in X} U_{mp}(x, f).$$  \hspace{1cm} (14)

Similar to Alg. 1, we solve this optimization problem, and then search for the optimal prices for different periods. However, one issue for this approach is that the pure strategy space $S$ grows exponentially large with the number $n$ of periods. E.g., when $n = 18$, $n_o = 10$, and $n_c = 4$, $|S|$ will be 176178. There will be as many variables in the above optimization problem, which makes the approach inapplicable to more fine grained discretization of periods.

### 3.2 Compact Representation of Mixed Strategies

We use a compact representation of a taxi driver’s mixed strategy to address the scalability issue. Such technique has been used to solve large scale security games (e.g.,[Kiekintveld et al., 2009], [An et al., 2011], [An et al., 2012], [Yin et al., 2012]). The idea is to represent each schedule with a set of Atom Schedules ($AS$), which has a much smaller size than $|S|$. We seek an optimal allocation of these $AS$ which maximizes $U_{mp}$, rather than an optimal mixed strategy as we do in Eq. (14). Using this optimal allocation of $AS$, we then construct a mixed strategy which maximizes $U_{mp}$, such that a solution to Eq. (14) is obtained.

Denote an $AS$ as a tuple $o(j, k)$. Following $o(j, k)$, a taxi driver works continuously from period $j$ to $k$. Let $O = \{o(j, k) : 1 \leq j, k \leq n, 0 \leq k - j < n_w\}$. Thus any schedule satisfying both C1 and C2 can be constructed by linking several $AS$ in $O$ with at least one resting periods between each pair of neighboring $AS$s, and there are less than $n_e \cdot |AS|$ in $O$, which is far less than $|S|$. We assign a weight $w_o$ to each $o \in O$ to represent the percentage of drivers whose schedule contains $o$. Denote the weights of all $o \in O$ as a vector $w$. $w$ determines an allocation of $AS$s. Therefore, the percentage $p$ of working taxis could be computed by $p^i = \sum_{o \in O} w_o \cdot \delta(o, i)$, where $\delta(o, i) = 1$ if period $i$ is covered by $o$, and $\delta(o, i) = 0$, otherwise.

Therefore, given $f$, $w$ determines $p$, and then $U_{mp}$. We formulate an optimization program as shown in Eqs. (15) to (18), which takes $w$ as variable and maximizes $U_{mp}$ for given $f$. Eqs. (16) to (18) restrict $w$ to a feasible domain, which ensures that a mixed strategy satisfying C1 and C2 can be constructed in the next step.

$$\max_{w} U_{mp}(w, f),$$  \hspace{1cm} (15)

subject to:

$$0 \leq w_o \leq 1, \quad \forall o \in O,$$  \hspace{1cm} (16)

$$0 \leq p^i + q^i \leq 1, \quad \forall i = 1, ..., n,$$  \hspace{1cm} (17)

$$\sum_{i=1}^{n} \sum_{o \in O} w_o \cdot \delta(o, i) \leq n_w.$$  \hspace{1cm} (18)

Here in Eq. (17), $q^i = \sum_{o \in O} w_o \cdot \delta'(o, i)$, where $\delta'(o, i) = 1$ if $o$ ends at period $i - 1$, and $\delta'(o, i) = 0$, otherwise. In other words, $q^i$ represents the percentage of drivers switching from work to rest at period $i - 1$, whom should not work in period $i$. Eq. (18) requires the overall coverage to be no larger than $n_w$.

As is justified by Proposition 1, Alg. 2 construct a mixed strategy satisfying C2, but not necessarily C1. The time complexity of Alg. 2 is $O(|O|^2)$.

**Proposition 1.** Alg. 2 returns a mixed strategy (a distribution over a set $A$ of schedules) which satisfies C2.

**Proof.** Firstly, each schedule in $A$ satisfies C2, because they are constructed by linking $AS$s leaving at least one resting period between each neighboring $AS$s (Line 6).

Next, we show that probabilities of all schedules in $A$ sum to 1, which is required by the definition of a mixed strategy. Let the $k$-th added schedule be $s_k$, and its probability be $x_{s_k}$. Since Lines 16 to 19 adds a special schedule with no periods covered, the sum of probabilities will always be no less than 1. Thus, we only need to show that $\sum_{i=1}^{|A|} x_{s_i} \leq 1$, where $|A|$ is the number of schedules in $A$. We prove this result by the following contradiction.

For $k = 1$, we have $\sum_{i=1}^{k} x_{s_i} = x_{s_1} \leq 1$. For some $k > 1$, suppose that $\sum_{i=1}^{k} x_{s_i} \leq 1$, but $\sum_{i=1}^{k+1} x_{s_i} > 1$.

Lines 5 to 7 search for the earliest appendable $AS$ for the schedule being constructed. Let $o(i, j)$ be the first $AS$ of $s_{k+1}$. This portion of $o(i, j)$, which is not used in the construction of $s_1, ..., s_k$, indicates that in each one of $s_1, ..., s_k$, at least one of period $i$ and period $i - 1$ is already covered. This is because assume that both of them are not covered in a
Algorithm 2: Construct a mixed strategy satisfying C2

1 $A \leftarrow \emptyset$, $SumP \leftarrow 0$;
2 $O' \leftarrow \{o(i, j) \mid o(i, j) \in O\}$ sorted by $i$ in ascending order;
3 while $O' \neq \emptyset$ do
4     $s \leftarrow$ create a null schedule, $k \leftarrow 0$;
5     for each $o(i, j) \in O'$ in ascending order do
6         if $i > k + 1$ then $k \leftarrow j$. Append $o$ to $s$;
7     end
8     $w_{\text{min}} \leftarrow$ minimum weight of $AS$s in $s$;
9     for each $o(i, j) \in s$ do
10        $w_o \leftarrow w_o - w_{\text{min}}$;
11        if $w_o = 0$ then Remove $o(i, j)$ from $O'$;
12    end
13    $x_s \leftarrow w_{\text{min}}$, add $s$ to $A$;
14    $SumP \leftarrow SumP + w_{\text{min}}$;
15 end
16 if $SumP < 1$ then
17     $s \leftarrow$ create a null schedule;
18     $x_s \leftarrow 1 - SumP$, add $s$ to $A$;
19 end
20 return $A$ with a probability $x_s$ for each $s \in A$;

schedule $s \in \{s_1, ..., s_k\}$, then $o(i, j)$ should be the earliest appendable $AS$ for $s$ after period $i - 1$, and should be appended to $s$ while $s$ is being constructed; thus period $i$ should be covered, which contradicts the assumption that it is not covered.

So, there exists an $AS$ $o^*(i^*, j^*)$ in each one of $s_1, ..., s_k$, which either covers period $i$ or ends at period $i - 1$. We have $\delta(o^*, i) + \delta'(o^*, i) = 1$ by the definition of $\delta$ and $\delta'$ in Section 3.2. Let $O^* = \{o^*(i^*, j^*) \mid o^* \in O, \delta(o^*, i) + \delta'(o^*, i) = 1\}$. For period $i$, we have

$$\sum_{o \in O} w_o [\delta(o, i) + \delta'(o, i)] \geq \sum_{o \in O^*} w_o \geq \sum_{o \in O^*} w_o.$$  

Furthermore, $w_o$ should be no less than the sum of probabilities of schedules in which $o$ appears. Since there exists an $AS$ $o \in O^*$ in each one of $s_1, ..., s_{k-1}$, we have

$$\sum_{o \in O} w_o [\delta(o, i) + \delta'(o, i)] \geq \sum_{o \in O^*} w_o \geq \sum_{o \in O^*} w_o,$$

which contradicts Eq. (17).

Therefore, we conclude that $\sum_{l=1}^{k} x_{s_l} \leq 1$ holds for any $k \geq 1$. Specially, we have $\sum_{l=1}^{k} x_{s_l} \leq 1$. We conclude that Alg. 2 returns a mixed strategy satisfying C2. □

Alg. 3 further adjusts the mixed strategy obtained by Alg. 2 to one that satisfies both C1 and C2. The core of Alg. 3 is a swapping method presented in Lines 5 to 7. Denote the number of periods covered by $s$ as $c(s)$. We find that if there is a schedule $s \in A$ which violates C1, i.e., $c(s) > n_w$, we can always find another schedule $s'$ where $c(s') < n_w$, because the overall coverage is no more than $n_w$ by Eq. (18).

We use a split to divide a schedule into two sub schedules. Denote a sub schedule of schedule $s$ as $s_j, k$, which begins at period $j$ and ends at period $k$. A split can be placed between two adjacent periods. Let the split right after period $i$ be split $i$. By applying split $i$ to $s$ and $s'$ such that $c(s) > n_w$ and $c(s') < n_w$, we divide them into sub schedules $s_1, s_i, s_{i+1}, n$, and $s_1', s_i', s_{i+1}'$ respectively. Then we execute a swapping procedure which links $s_1, s_i$ to $s_1', s_i', s_{i+1}$ to create two new schedules denoted as $s_1, s_i, s_{i+1}, n$ and $s_1', s_i', s_{i+1}'$. Proposition 2 shows that we can always find a safe split $i$, such that $c(s_1, i, s_{i+1}, n) = n_w$. Here a split is safe if a swapping on it does not cause violation of C2.

**Proposition 2.** For schedules $s$ and $s'$, where $c(s) > n_w$ and $c(s') < n_w$, a safe split $i$ with $c(s_1, i, s_{i+1}, n) = n_w$ can always be found.

**Proof.** We search for such a split with the loop shown in Lines 5 and 7 of Alg. 3, and prove that this loop always ends with a safe split $i$, such that $c(s_1, i, s_{i+1}, n) = n_w$. We first show that this loop will never encounter a split such that $c(s_1, i, s_{i+1}, n) > n_w + 1$.

The for-loop in Lines 5 to 7 checks if the split in the current iteration is safe and if it covers $n_w$ periods. Assume that the for-loop encounters a split $i$, such that $c(s_1, i, s_{i+1}, n) = n_w + 1$ for the first time. We find that $c(s_1, i, s_{i+1}, n) = n_w$ because $c(s_1, i, s_{i+1}, n) - c(s_1, i-1, s_{i+1}, n) = s_i - s_i' \in \{1, 0\}$, and $c(s_1, i-1, s_{i+1}, n)$ cannot be $n_w + 1$ or $n_w + 2$ since we assume that split $i$ is the first split such that $c(s_1, i, s_{i+1}, n) = n_w + 1$. Therefore, we have $c(s_1, i, s_{i+1}, n) = n_w$, and $s_i = 1, s_i' = 0$. Consider the two periods right before split $i$, Fig. 3 depicts all possible covering patterns. However, all these patterns will in fact not appear:

- Clearly, for patterns shown in Figs. 3(a) and 3(b), split $i - 1$ is safe, together with the fact that $c(s_1, i-1, s_{i+1}, n) = n_w$, the loop should end before it encounters split $i$. Therefore, these two patterns will not appear.
For pattern shown in Fig. 3(c), we find that \( c(s_{1,i-2} + s'_{1,i-1}) = n_w + 1 \), which contradicts the assumption that split \( i \) is the first such split. Therefore, this pattern will not appear.

For pattern shown in Fig. 3(d), we find split \( i - 1 \) is also safe. Because if it is not safe, then the AS in \( s' \) ending at period \( i - 1 \) must start earlier than the AS in \( s \) which covers period \( i \). Let \( j \) be the index of the starting period of the AS in \( s' \) which covers period \( i \), we have \( c(s_{1,j-2} + s'_{1,j-1}) = n_w + 1 \) as shown in Fig. 4, which contradicts the assumption that split \( i \) is the first split such that \( c(s_{1,i} + s'_{i+1,n}) \geq n_w + 1 \). Therefore, this pattern will not appear.

Therefore, the for-loop will never encounter a split \( i \), such that \( c(s_{1,i} + s'_{i+1,n}) = n_w + 1 \). Moreover, since \( c(s_{1,i} + s'_{i+1,n}) \) grows by at most 1 in each iteration of increasing \( i \) by 1, it can not jump from a value smaller than \( n_w + 1 \) to another one larger than \( n_w + 1 \). Thus the loop will in fact never encounter a split \( i \), such that \( c(s_{1,i} + s'_{i+1,n}) \geq n_w + 1 \).

Since we already have that when \( i = n \), \( c(s_{1,n} + s'_{n+1,n}) = c(s_{1,n} + s'_{n+1,n}) = c(s) \geq n_w + 1 \). We conclude that the loop will always end before \( i \) reaches \( n \), thus the only reason for its ending is that it breaks with a safe split \( i \) such that \( c(s_{1,i} + s'_{i+1,n}) = n_w \) (Line 6).

Thus we transform \( s \) and \( s' \) into two new schedules with coverages of \( n_w \) and \( c(s) + c(s') - n_w \) respectively. For the latter one, we have \( c(s) + c(s') - n_w < c(s) \), which is, if not smaller or equal to \( n_w \), at least one period closer to \( n_w \) than \( c(s) \). Note that we only swap \( s \) and \( s' \) by a feasible portion. For example, if the probabilities of \( s \) and \( s' \) are 0.5 and 0.3, we can only take a portion of 0.3 from \( s \) to swap with \( s' \). The maximum swapable portion of \( s \) and \( s' \) is the minimum of their probabilities (Line 9). The left portion will be handled by swapping with another \( s' \) as Alg. 3 proceeds.

We show that Alg. 3 will eventually end after swapping for at most \( 2^n \) times. In Line 1, we divide \( A \) into subsets \( A_k \). After each swapping, at least one of \( s \) and \( s' \) will be removed (Lines 12 and 13), and at least one of \( |A_{c(s)}| \) and \( |A_{c(s')}| \) will be reduced by 1. The newly added schedules (Line 10) are always added to a subset \( A_k \), such that \( c(s') < k < c(s) \), which will never increase \( |A_{c(s)}| \) or \( |A_{c(s')}| \). Since \( |A_k| \leq (\frac{n}{k}) \), after swapping for at most \( \sum_{k=0}^{n} (\frac{n}{k}) = 2^n \) times, all the subsets except \( A_{n_w} \) will be empty, and Alg. 3 will thus end.

In addition, swapping does not change the value of \( p' \), such that Eq. (18) can always be satisfied. For Eq. (17), since in each schedule, there can be at most one \( AS \) which covers period \( i \), or ends at period \( i-1 \), we have \( p' + q' \leq \sum_{s \in A} w_s \leq 1 \), which means Eq. (17) will always be satisfied.

### 3.3 Optimal Price for Multiple Periods

We extend the objective function for a single period to one for multiple periods by summing up the objective functions of all periods. Alg. 4 searches from a set of candidate prices for each period. For each price scheme, Alg. 4 solves the program presented in Eqs. (15) to (18) to obtain each taxi driver’s optimal strategy. Here we transform this program to another form as shown in Eqs. (19) to (20) by taking \( D \) and \( W \) as independent variables, so as to address the implicitly defined relationship between them.

**Algorithm 3: Adjust a mixed strategy to satisfy C1, C2**

1. \( A_k = \{ s : s \in A, c(s) = k \}, k = 1, ..., n \)
2. while \( \bigcup_{k=n_w+1}^{n} A_k \neq \emptyset \) do
   3. Let \( s \) be the schedule with highest \( c(s) \) in \( \bigcup_{k=n_w+1}^{n} A_k \);
   4. Let \( s' \) be the schedule with lowest \( c(s') \) in \( \bigcup_{k=1}^{n} A_k \);
   5. for \( i = 1 \) to \( n \) do
      6. if \( c(s_{1,i} + s'_{i+1,n}) = n_w \) and split \( i \) is safe then Break;
   7. end
   8. \( t \leftarrow s_{1,i} + s'_{i+1,n}, t' \leftarrow s_{1,i} + s_{i+1,n} \);
   9. \( x \leftarrow \min\{x_s, x_{s'}\} \);
   10. Add \( t \) to \( A_{n_w} \); Add \( t' \) to \( A_{c(t')} \);
   11. \( x_s \leftarrow x_{s'} = x', x_{s'} \leftarrow x' - x; \)
   12. if \( x = 0 \) then remove \( s \) from \( A_{c(s)} \);
   13. if \( x_{s'} = 0 \) then remove \( s' \) from \( A_{c(s')} \);
14. end
15. return \( \bigcup_{k=1}^{n_w} A_k \) with probability \( x_s \) for each \( s \in \bigcup_{k=1}^{n_w} A_k \);

\[
\max_{w,D,W} U_{mp} = \sum_{i=1}^{n} \frac{D_i F_i}{mT} - c_g t \sum_{s \in A} w(s) \cdot \delta(s, i),
\]

**s.t.**

\[
D_i - D_{i+1} \leq \beta (\phi_i W + \phi_2 L + F) = 0, \quad \forall i = 1, ..., n
\]

\[
W_i - \omega (T_w - \frac{p_i d}{m V_i / t}) = 0, \quad \forall i = 1, ..., n
\]

\[
0 \leq w_i \leq 1, \quad \forall i \in \mathcal{O}
\]

\[
0 \leq p' + q' \leq 1, \quad \forall i = 1, ..., n
\]

\[
\sum_{i=1}^{n} \sum_{o \in \mathcal{O}} w_o \cdot \delta(o, i) \leq n_w
\]

\[
D_i, W_i > 0, \quad \forall i = 1, ..., n
\]

Rather than optimizing \( f \) as an \( n \)-dimensional vector, we reduce it to two segments: one for normal periods, and the other for peak periods. The reason that we propose such a simplified fare structure for multiple periods is that firstly, research (e.g., [Bilton, 2012]) has shown that a too complex fare structure may make customers unpleasant, as the total fare will be hard to predict; secondly, with such a simplified fare
structure, the computational complexity could be largely reduced. We use the single-period model to calculate customer demand with variance of per unit distance fare for each period. We identify periods where customer demand decreases at prices larger than ¥2.00 as normal periods, and the rest as peak periods. This is because, intuitively, customers are more sensitive to the fare price at normal periods when taxi supply is sufficient, such that extra incentive for these periods could hardly attract more taxi drivers to work. Next, Alg. 4 fixes the fare price for normal periods to ¥2.00, and searches the optimal fare price for peak periods within the candidates. We can also relax the price for normal periods, and search on a two-dimensional space to optimize prices for both peak and normal periods, for which we just need more iterations.

4 Experimental Evaluation

Experiments based on real data are provided to evaluate our approach. We use software KNITRO (version 8.0.0) to solve all the non-linear programs, e.g., Eqs. (11) and (12).

4.1 Data Sets

According to Annual Report on Beijing's Transportation 2010, the total number of Beijing's licensed taxis $T = 6.66 \times 10^4$, average trip distance $d = 7.2$ km. Suppose $D_{max}$ for each period is proportional to the total demand for bus, subway and taxi services, and $N_{nor}$ for each period is proportional to the demand of private cars. We set values of $D_{max}$ and $N_{nor}$ as depicted in Table 1. We take 1 hour as a period, i.e., $\tau = 1$ hour, and consider 18 periods from 5:00 to 23:00, since customer demand in other time is negligible. We set maximum working time $n_w = 9$ hours, and maximum continuous working time $n_c = 4$ hours.

Table 1: Period-dependent parameters

<table>
<thead>
<tr>
<th>$i$</th>
<th>Begin time</th>
<th>$D_{max}^i$ ($10^3$)</th>
<th>$N_{nor}^i$ ($10^3$)</th>
<th>$i$</th>
<th>Begin time</th>
<th>$D_{max}^i$ ($10^3$)</th>
<th>$N_{nor}^i$ ($10^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>05:00</td>
<td>8.99</td>
<td>8.53</td>
<td>10</td>
<td>14:00</td>
<td>50.46</td>
<td>46.38</td>
</tr>
<tr>
<td>2</td>
<td>06:00</td>
<td>49.43</td>
<td>54.64</td>
<td>11</td>
<td>15:00</td>
<td>63.72</td>
<td>54.96</td>
</tr>
<tr>
<td>3</td>
<td>07:00</td>
<td>151.93</td>
<td>81.75</td>
<td>12</td>
<td>16:00</td>
<td>52.90</td>
<td>59.48</td>
</tr>
<tr>
<td>4</td>
<td>08:00</td>
<td>106.87</td>
<td>78.14</td>
<td>13</td>
<td>17:00</td>
<td>159.44</td>
<td>81.53</td>
</tr>
<tr>
<td>5</td>
<td>09:00</td>
<td>55.42</td>
<td>53.34</td>
<td>14</td>
<td>18:00</td>
<td>112.40</td>
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<tr>
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<td>15</td>
<td>19:00</td>
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<td>59.53</td>
<td>16</td>
<td>20:00</td>
<td>45.96</td>
<td>47.95</td>
</tr>
<tr>
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<td>12:00</td>
<td>49.27</td>
<td>47.14</td>
<td>17</td>
<td>21:00</td>
<td>28.88</td>
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</tr>
<tr>
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<td>13:00</td>
<td>41.45</td>
<td>43.13</td>
<td>18</td>
<td>22:00</td>
<td>21.13</td>
<td>21.99</td>
</tr>
</tbody>
</table>

For the function of travel speed and number of working taxis defined in Eq. (2), we set $v_f = 50$ km/h, $N_{max} = 100 \times 10^3$, thus we have $V^i (p^i) = -5.00 \times 10^{-5} (N_{nor}^i + 6.66 \times 10^4 p^i) + 50.00$. We set gas price $c_g = ¥20$/hour from interviews to taxi drivers. We set $\beta = 0.06$, $\omega = 400$ with reference to researches from Yang et al. (e.g., [2005b]), and set $m = 1.5$ according to media reports. We set $\phi_1 = ¥40$/hour and $\phi_2 = ¥40$/hour given that: 1) $\phi_1$ and $\phi_2$ should be similar since they are both values for time; 2) $\phi_2$ should be larger than $\phi_1$ since, empirically, waiting time is more critical on customer’s experience to taxi service.

4.2 Experiment Results for a Single Period

We use the single-period model to pick out peak periods. Customer demands at prices varying from ¥1.00 to ¥8.00 with a difference of ¥0.50 are calculated for each period. Experiment results for the 7th period (11:00 - 12:00), and the 13th period (17:00 - 18:00) are presented in Fig. 5. As is shown in Fig. 5(a), customer demand decreases with the price in the peak period, while at the same time customer demand of the normal periods are improved compared with the current situation (Fig. 6(a)), while at the same time customer demand of the whole day increases from $187.78 \times 10^4$ to $200.50 \times 10^4$. Fig 6(b) shows two significant valleys of the number of working taxis defined in Eq. (2), we set $v_f = 50$ km/h, $N_{max} = 100 \times 10^3$, thus we have $V^i (p^i) = -5.00 \times 10^{-5} (N_{nor}^i + 6.66 \times 10^4 p^i) + 50.00$. We set gas price $c_g = ¥20$/hour from interviews to taxi drivers. We set $\beta = 0.06$, $\omega = 400$ with reference to researches from Yang et al. (e.g., [2005b]), and set $m = 1.5$ according to media reports. We set $\phi_1 = ¥20$/hour and $\phi_2 = ¥40$/hour given that: 1) $\phi_1$ and $\phi_2$ should be similar since they are both values for time; 2) $\phi_2$ should be larger than $\phi_1$ since, empirically, waiting time is more critical on customer’s experience to taxi service.

4.3 Experiment Results for Multiple Periods

The optimal price for peak periods obtained using the multi-period model is ¥3.00. Under this price, customer demands of peak periods are improved compared with the current situation (Fig. 6(a)), while at the same time customer demand of the whole day increases from $187.78 \times 10^4$ to $200.50 \times 10^4$. Fig 6(b) shows two significant valleys of the number of working taxis in the peak periods in the current market, which accords with the reality. Under the optimal price, these two valleys both move upward, which indicates the effectiveness of the extra incentive. Percentages of taxis in the normal periods exhibit the similar patterns, because fixed prices for these periods result in the same market conditions of each period. However, there are some slight changes over these
periods caused by the interdependence of taxi drivers’ strategies among different periods. Fig. 6(c) depicts variance of a taxi driver’s expected revenues. Fig. 6(d) depicts total working time of taxi drivers under different prices. The drivers are more willing to work with higher prices, however, the total working time is bounded after the price is higher than ¥3.00, where the maximum working time is reached.

5 Conclusions
The key contributions of this paper include: (1) a realistic taxi market model considering taxi drivers’ behavior under market variance over multiple periods; (2) an atom schedule method with novel algorithms to address computational challenges due to the large size of pure strategies; and (3) experiments based on real data which show that the new pricing scheme achieved significant efficiency improvement. We believe this study will stimulate the interest in applying innovative AI techniques to improve taxi system efficiency and other sustainability issues in megacities.

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References